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Twisted inner products and contraction inequalities on spaces of contravariant and covariant tensors

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Abstract

Given positive integers n and p, and a complex finite dimensional vector space V, we let $\mathbf{S}_{n,p}(V)$ denote the set of all functions from $V \times V \times \cdots \times V$-$(n+p$ copies) to $\mathbb{C}$ that are linear and symmetric in the first n positions, and conjugate linear symmetric in the last p positions. Letting $\kappa = \min\{n, p\}$ we introduce twisted inner products, $[\cdot, \cdot]_{s,t}$, $1 \leqslant s, t \leqslant \kappa$, on $\mathbf{S}_{n,p}(V)$, and prove the monotonicity condition $[F, F]_{s,t} \geqslant [F, F]_{u,v}$ is satisfied when $s \leqslant u \leqslant \kappa$, $t \leqslant v \leqslant \kappa$, and $F \in \mathbf{S}_{n,p}(V)$. Using the monotonicity condition, and the Cauchy–Schwartz inequality, we obtain as corollaries many known inequalities involving norms of symmetric multilinear functions, which in turn imply known inequalities involving permanents of positive semidefinite Hermitian matrices. New tensor and permanental inequalities are also presented. Applications to Partial Differential Equations are indicated.

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1. Introduction

Let V be a complex vector space of dimension m with inner product $\langle\cdot, \cdot\rangle$, and orthonormal basis $\{e_i\}_{i=1}^m$. By V^* we mean the dual of V, and if $x \in V$, then $\tilde{x}$, a member of V^*, is defined by $\tilde{x}(y) = \langle y, x\rangle$ for all $y \in V$. The inner product $\langle\cdot, \cdot\rangle$ extends to V^* in the usual way, and, with respect to the conjugate linear bijection $w \mapsto \tilde{w}$, we have $\langle\tilde{x}, \tilde{y}\rangle = \langle y, x\rangle$ for all $x, y \in V$. For

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positive integers u and v, $\mathbf{T}_{u,v}(V)$ denotes the vector space whose elements are the functions from $V \times V \times \cdots \times V$-($u+v$ copies) to $\mathbb{C}$ that are linear in the first u positions, and conjugate linear in the last v positions. Essentially, $\mathbf{T}_{u,v}(V)$ is the set of all tensors that are u-covariant and v-contravariant over V. Then, $\mathbf{T}_{u,0}(V)$, denoted by $\mathbf{T}_u(V)$, is the set of all u-linear $\mathbb{C}$-valued functions on $V \times V \times \cdots \times V$-($u$-copies), and $\mathbf{T}_{0,v}(V)$ is the set of all v-conjugate linear $\mathbb{C}$-valued functions defined on $V \times V \times \cdots \times V$-($v$-copies). Of course, $\mathbf{T}_1(V) = V^*$, and we identify $\mathbf{T}_{0,0}(V)$ with $\mathbb{C}$.

If $F \in \mathbf{T}_{u,v}(V)$, $0 \leqslant s \leqslant u$, and $0 \leqslant t \leqslant v$, then the insertion $F(x_1, x_2, \ldots, x_s; y_1, y_2, \ldots, y_t)$, where $x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_t \in V$, is the member of $\mathbf{T}_{u-s,v-t}(V)$ such that

$$\begin{aligned} &F(x_1, x_2, \ldots, x_s; y_1, y_2, \ldots, y_t)(z_1, z_2, \ldots, z_{u+v-s-t}) \\ &\quad = F(x_1, x_2, \ldots, x_s, z_1, z_2, \ldots, z_{u-s}; y_1, y_2, \ldots, y_t, z_{u-s+1}, z_{u-s+1}, \ldots, z_{u+v-s-t}) \end{aligned} \tag{1}$$

for all $z_1, z_2, \ldots, z_{u+v-s-t} \in V$. By default, definition (1) includes the case of insertions into members of $\mathbf{T}_u(V)$. Generally $\Gamma_{s,t}$, denotes set of all functions from $\{1, 2, \ldots, s\}$ to $\{1, 2, \ldots, t\}$, but when t is m, the dimension of V, we simply write Γ_s. If $G \in \mathbf{S}_u(V)$, $s \leqslant u$, and $f \in \Gamma_s$, then $G(e_f)$, which we further abbreviate to G_f, denotes the insertion $G(e_{f(1)}, e_{f(2)}, \ldots, e_{f(s)})$; moreover, if $H \in \mathbf{T}_{u,v}(V)$, then $H(e_f; e_g)$, alternately $H_{f,g}$, denotes $H(e_{f(1)}, e_{f(2)}, \ldots, e_{f(s)}; e_{g(1)}, e_{g(2)}, \ldots, e_{g(t)})$ for all $f \in \Gamma_s$, and $g \in \Gamma_t$.

We transform the complex vector spaces $\mathbf{T}_{u,v}(V)$ into inner product spaces by defining

$$\langle A, B\rangle = \sum_{f\in\Gamma_u}\sum_{g\in\Gamma_v} A(e_f; e_g)\overline{B(e_f; e_g)} \quad \forall A, B \in \mathbf{T}_{u,v}(V). \tag{2}$$

That the inner product $\langle\cdot,\cdot\rangle$ on $\mathbf{T}_{u,v}(V)$ is independent of $\{e_i\}_{i=1}^m$ is easy to establish, but follows nevertheless from [2, Theorem 4.5]. By $\mathbf{S}_{u,v}(V)$ we mean the subspace of $\mathbf{T}_{u,v}(V)$ consisting of the elements of $\mathbf{T}_{u,v}(V)$ that are symmetric in the first u positions and in the last v positions. If $v = 0$, then $\mathbf{S}_{u,v}(V)$ is denoted by $\mathbf{S}_u(V)$, and is the set of all fully symmetric $F \in \mathbf{T}_u(V)$.

Fundamental to our presentation are the linear contraction maps $\mathbf{C}_t : \mathbf{S}_{u,v}(V) \to \mathbf{S}_{u-t,v-t}(V)$ defined by

$$\mathbf{C}_t(F) = \sum_{\phi\in\Gamma_t} F(e_\phi; e_\phi) = \sum_{\phi\in\Gamma_t} F_{\phi,\phi} \quad \forall F \in \mathbf{S}_{u,v}(V),\ 1 \leqslant t \leqslant \min\{u, v\}. \tag{3}$$

By $\mathbf{C}_0$ we mean the identity map on $\mathbf{S}_{u,v}(V)$. Explicitly, if $x_1, x_2, \ldots, x_{u-t}, y_1, y_2, \ldots, y_{v-t} \in V$, then

$$\begin{aligned} &\mathbf{C}_t(F)(x_1, x_2, \ldots, x_{u-t}; y_1, y_2, \ldots, y_{v-t}) \\ &\quad = \sum_{\phi\in\Gamma_t} F(e_\phi, x_1, x_2, \ldots, x_{u-t}; e_\phi, y_1, y_2, \ldots, y_{v-t}) \\ &\quad = \sum_{\phi\in\Gamma_t} F(e_{\phi(1)}, e_{\phi(2)}, \ldots, e_{\phi(t)}, x_1, x_2, \ldots, x_{u-t}; e_{\phi(1)}, e_{\phi(2)}, \ldots, e_{\phi(t)}, y_1, y_2, \ldots, y_{v-t}). \end{aligned} \tag{4}$$

Like $\langle\cdot,\cdot\rangle$, the $\mathbf{C}_t$ are independent of $\{e_i\}_{i=1}^m$; moreover, $\mathbf{C}_s \circ \mathbf{C}_t = \mathbf{C}_{s+t}$ when $s + t \leqslant \min\{u, v\}$.

Related to the $\mathbf{C}_t$, and important to our presentation, are contractions $\bar{\otimes}_t$, introduced by Neuberger [4] in connection with Neuberger's lower bound inequality for the norm of the symmetric product. For each $A \in \mathbf{S}_u(V)$, $B \in \mathbf{S}_v(V)$, and t such that $0 \leqslant t \leqslant \min\{u, v\}$ we set

$$(A \bar{\otimes}_t B)(x_1, x_2, \ldots, x_{u-t}, y_1, y_2, \ldots, y_{v-t}) = \langle A(x_1, x_2, \ldots, x_{u-t}), B(y_1, y_2, \ldots, y_{v-t})\rangle \tag{5}$$

for all $x_1, x_2, \ldots, x_{u-t}, y_1, y_2, \ldots, y_{v-t} \in V$. By $A \bar{\otimes} B$ we mean $A \bar{\otimes}_0 B$. The maps $\mathbf{C}_t$ are the natural extensions of the contractions $\bar{\otimes}_t$, for if $A \in \mathbf{S}_u(V)$ and $B \in \mathbf{S}_v(V)$, then (3) and (5) imply that

$$A \bar{\otimes}_t B = \sum_{\phi \in \Gamma_t} A(e_\phi) \bar{\otimes} B(e_\phi) = \sum_{\phi \in \Gamma_t} (A \bar{\otimes} B)(e_\phi; e_\phi) = \mathbf{C}_t(A \bar{\otimes} B). \tag{6}$$

The distinction between $A \bar{\otimes} B$ and the standard tensor product $A \otimes B$ defined by

$$(A \otimes B)(x_1, x_2, \ldots, x_{u+v}) = A(x_1, x_2, \ldots, x_u)B(x_{u+1}, x_{u+2}, \ldots, x_{u+v})$$

for all $x_1, x_2, \ldots, x_{u+v}$ in V should be noted. Nevertheless, both $A \otimes B$ and $A \bar{\otimes} B$ have the same norm, as it is evident that $\|A \otimes B\|^2 = \|A \bar{\otimes} B\|^2 = \|A\|^2\|B\|^2$ for all $A \in \mathbf{S}_u(V)$ and $B \in \mathbf{S}_v(V)$.

If $\sigma \in \mathbb{S}_u$, the symmetric group on $\{1, 2, \ldots, u\}$, and $A \in \mathbf{T}_u(V)$, then the product σA is defined by $(\sigma A)(x_1, x_2, \ldots, x_u) = A(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(u)})$ for all $x_1, x_2, \ldots, x_u \in V$. Thus, $\sigma A \in \mathbf{T}_u(V)$ for all $A \in \mathbf{T}_u(V)$ and $\sigma \in \mathbb{S}_u$, and the mapping $(\sigma, A) \mapsto \sigma A$ is a group action of $\mathbb{S}_u$ on $\mathbf{T}_u(V)$. Furthermore, $\mathbf{S}_u(V)$ is the set of all $A \in \mathbf{T}_u(V)$ such that $\sigma A = A$ for all $\sigma \in \mathbb{S}_u$. The action of $\mathbb{S}_u$ on $\mathbf{T}_u(V)$ extends linearly to the algebra of functions from $\mathbb{S}_u$ to $\mathbb{C}$, denoted by $\mathbb{C}\mathbb{S}_u$; thus, if $f : \mathbb{S}_u \to \mathbb{C}$, and we represent f in formal sum notation as $\sum_{\sigma \in \mathbb{S}_u} f(\sigma)\sigma$, then for $A \in \mathbf{T}_u(V)$ we define fA by $fA = \sum_{\sigma \in \mathbb{S}_u} f(\sigma)\sigma A$. If $f \in \mathbb{C}\mathbb{S}_u$, then f^* denotes $\sum_{\sigma \in \mathbb{S}_u} \overline{f(\sigma)}\sigma$; thus, f^* is also a member of $\mathbb{C}\mathbb{S}_u$, and $f^*(\sigma) = \overline{f(\sigma)}$ for all $\sigma \in \mathbb{S}_u$. Elements satisfying $f^* = f$ are said to be Hermitian. Note $\langle fA, B\rangle = \langle A, f^*B\rangle$ for all $f \in \mathbb{C}\mathbb{S}_u$, and $A, B \in \mathbf{T}_u(V)$.

The symmetrizer $\mathscr{P}_u$ defined by $\mathscr{P}_u = (u!)^{-1} \sum_{\sigma \in \mathbb{S}_u} \sigma$ is real-valued and hence Hermitian. Moreover, $\sigma\mathscr{P}_u = \mathscr{P}_u\sigma = \mathscr{P}_u$ for all $\sigma \in \mathbb{S}_u$; so $\mathscr{P}_u^2 = \mathscr{P}_u$. Thus, $\mathscr{P}_u$ is a Hermitian idempotent in $\mathbb{C}\mathbb{S}_u$ that commutes with the members of $\mathbb{S}_u$. Since $\mathscr{P}_u^* = \mathscr{P}_u$, we have $\langle \mathscr{P}_u A, B\rangle = \langle A, \mathscr{P}_u^* B\rangle = \langle A, \mathscr{P}_u B\rangle$. Regarded as a linear map on $\mathbf{T}_u(V)$, $\mathscr{P}_u$ is therefore the orthogonal projection of $\mathbf{T}_u(V)$ onto $\mathbf{S}_u(V)$.

If $A \in \mathbf{T}_u(V)$ and $B \in \mathbf{T}_v(V)$, then the symmetric product $A \cdot B$ is $\mathscr{P}_{u+v}(A \otimes B)$. Lower bound estimates for norms of symmetric products have proved to be important for proving convergence theorems for iterative procedures for the solutions of partial differential equations. Indeed, this was the present author's initial reason for pursuing inequalities of type presented herein. Moreover, there are many results similar to those presented in [8] and [9] that might be proved using inequalities like those we present here. Such work is in progress. Similar efforts in the area of Partial Differential Equalitions are presented by Neuberger in [5]. Such inequalities have also been useful in examining certain conjectured inequalities involving immanents, notably [10] and [13]. An important relationship between symmetric products $A \cdot B$ and the maps $\bar{\otimes}_t$, and hence the contractions $\mathbf{C}_t$, was established by Neuberger [4] who showed that if $A, C \in \mathbf{S}_n(V)$ and $B, D \in \mathbf{S}_p(V)$, then

$$\begin{aligned}\binom{n+p}{n} \langle A \cdot B, C \cdot D\rangle &= \sum_{t=0}^{\kappa} \binom{n}{t}\binom{p}{t} \langle A \bar{\otimes}_t D, C \bar{\otimes}_t B\rangle \\ &= \sum_{t=0}^{\kappa} \binom{n}{t}\binom{p}{t} \langle \mathbf{C}_t(A \bar{\otimes} D), \mathbf{C}_t(C \bar{\otimes} B)\rangle, \end{aligned} \tag{7}$$

where $\kappa = \min\{n, p\}$. The latter equality in (7), though not stated explicitly in [4], follows immediately from the first equality because of (6). Setting $C = A$ and $D = B$ in (7) we obtain the equality

$$\binom{n+p}{n} \|A \cdot B\|^2 = \sum_{t=0}^{\kappa} \binom{n}{t}\binom{p}{t} \|\mathbf{C}_t(A \bar{\otimes} B)\|^2. \tag{8}$$

Since $\|\mathbf{C}_0(A \bar{\otimes} B)\|^2 = \|A \bar{\otimes} B\|^2 = \|A\|^2\|B\|^2$, (8) immediately implies that

$$\binom{n+p}{n} \|A \cdot B\|^2 \geqslant \|A\|^2\|B\|^2 \tag{9}$$

for all $A \in \mathbf{S}_n(V)$ and $B \in \mathbf{S}_p(V)$. An extension of (7) presented in [12] is Theorem 1.

Theorem 1. *Suppose n, p, q, and r are positive integers such that $n + p = q + r$. If $A \in \mathbf{S}_n(V)$, $B \in \mathbf{S}_p(V)$, $C \in \mathbf{S}_q(V)$, and $D \in \mathbf{S}_r(V)$, then*

$$\binom{n+p}{q} \langle A \cdot B, C \cdot D\rangle = \sum_{s=\kappa_1}^{\kappa_2} \binom{n}{n-s}\binom{p}{q-s} \langle \mathbf{C}_{n-s}(A \bar{\otimes} D), \mathbf{C}_{q-s}(C \bar{\otimes} B)\rangle \tag{10}$$

where $\kappa_1 = \max\{0, n-r\} = \max\{0, q-p\}$, and $\kappa_2 = \min\{n, q\}$.

If $q = n$ and $r = p$, then $\kappa_1 = \max\{0, n-p\}$ and $\kappa_2 = n$; thus, (10) reduces to

$$\binom{n+p}{n} \langle A \cdot B, C \cdot D\rangle = \sum_{s=\kappa_1}^{n} \binom{n}{n-s}\binom{p}{n-s} \langle \mathbf{C}_{n-s}(A \bar{\otimes} D), \mathbf{C}_{n-s}(C \bar{\otimes} B)\rangle. \tag{11}$$

If we substitute $t = n - s$ in (11), and note that the new upper limit of summation, namely $n - \max\{0, n-p\}$, is the same as $\min\{n, p\}$, denoted by κ, then we obtain (7); thus, Theorem 1 includes (7) as a special case.

2. Twisted inner products

Let n and p be positive integers and let $\kappa = \min\{n, p\}$. For each integer pair (s, t) such that $0 \leqslant s, t \leqslant \kappa$ we will construct a special sesquilinear form $[\cdot, \cdot]_{s,t}$ on $\mathbf{S}_{n,p}(V)$ via linear extension. The primary result is Theorem 3 wherein we prove that each of the $[\cdot, \cdot]_{s,t}$ is an inner product on $\mathbf{S}_{n,p}(V)$, and that the monotonicity condition $[F, F]_{s,t} \geqslant [F, F]_{u,v}$ when $s \leqslant u$ and $t \leqslant v$ holds for all $F \in \mathbf{S}_{n,p}(V)$. In Section 3 we apply the monotonicity condition and the Cauchy-Schwartz inequality to easily obtain several inequalities, some known and some new, involving norms of symmetric multilinear functions. In Section 4 we specialize the inequalities of Section 3 to the case of fully decomposable symmetric multilinear functions thereby obtaining results that translate into inequalities involving the permanent function restricted to the set of positive semidefinite Hermitian matrices. All results presented in Sections 3 and 4 are regarded as being corollaries to Theorem 3.

For decomposable functions $A \bar{\otimes} B$ and $C \bar{\otimes} D$, where $A, C \in \mathbf{S}_n(V)$ and $B, D \in \mathbf{S}_p(V)$, and integers s and t such that $0 \leqslant s \leqslant \kappa$ and $0 \leqslant t \leqslant \kappa$, we define

$$[A \bar{\otimes} B, C \bar{\otimes} D]_{s,t} = \binom{n+p-s-t}{n-s} \sum_{f \in \Gamma_s} \sum_{g \in \Gamma_t} \langle A_f \cdot D_g, C_f \cdot B_g\rangle. \tag{12}$$

Any $H \in \mathbf{S}_{n,p}(V)$ is expressible as a sum of functions of the form $A \bar{\otimes} B$; thus, if $F, G \in \mathbf{S}_{n,p}(V)$, then we have $F = \sum_{i=1}^{u} A^i \bar{\otimes} B^i$, and $G = \sum_{j=1}^{v} C^j \bar{\otimes} D^j$ for some $A^1, A^2, \ldots, A^u, C^1,$

$C^2, \dots, C^v$ in $\mathbf{S}_n(V)$, and $B^1, B^2, \dots, B^u, D^1, D^2, \dots, D^v$ in $\mathbf{S}_p(V)$. Therefore, in accordance with the idea of linear extension, we set

$$[F, G]_{s,t} = \sum_{i=1}^{u}\sum_{j=1}^{v}[A^i \mathbin{\bar{\otimes}} B^i, C^j \mathbin{\bar{\otimes}} D^j]_{s,t}$$

$$= \binom{n+p-s-t}{n-s} \sum_{i=1}^{u}\sum_{j=1}^{v}\sum_{f\in\Gamma_s}\sum_{j\in\Gamma_t}\langle A^i_f \cdot D^j_g, C^j_f \cdot B^i_g\rangle. \tag{13}$$

If $s = 0$ or $t = 0$, then the corresponding summations are absent; in particular, we have

$$[A \mathbin{\bar{\otimes}} B, C \mathbin{\bar{\otimes}} D]_{0,0} = \binom{n+p}{n}\langle A \cdot D, C \cdot B\rangle. \tag{14}$$

We must prove that the $[\cdot,\cdot]_{s,t}$ are well-defined and positive-definite. That is the purpose of Theorem 2. Theorem 2 is another extension of Neuberger's identity, as it reduces to (7) when $u = v = 1$ and $s = t = 0$. Let $\mu_{s,t} = \min\{n-s, p-t\}$.

Theorem 2. *Suppose* $\{A^i\}_{i=1}^u \subset \mathbf{S}_n(V)$, $\{C^j\}_{j=1}^v \subset \mathbf{S}_n(V)$, $\{B^i\}_{i=1}^u \subset \mathbf{S}_p(V)$, *and* $\{D^j\}_{j=1}^v \subset \mathbf{S}_p(V)$. *If* $F = \sum_{i=1}^u A^i \mathbin{\bar{\otimes}} B^i$ *and* $G = \sum_{j=1}^v C^j \mathbin{\bar{\otimes}} D^j$, *then*

$$\binom{n+p-s-t}{n-s}\sum_{i=1}^{u}\sum_{j=1}^{v}\sum_{f\in\Gamma_s}\sum_{g\in\Gamma_t}\langle A^i_f \cdot D^j_g, C^j_f \cdot B^i_g\rangle$$

$$= \sum_{w=0}^{\mu_{s,t}}\binom{n-s}{w}\binom{p-t}{w}\langle \mathbf{C}_w(F), \mathbf{C}_w(G)\rangle \tag{15}$$

for all integers s and t such that $0 \leqslant s \leqslant \kappa$ *and* $0 \leqslant t \leqslant \kappa$.

Proof. If $1 \leqslant i \leqslant u$, $1 \leqslant j \leqslant v$, $f \in \Gamma_s$, and $g \in \Gamma_t$, then $A^i_f \in \mathbf{S}_{n-s}(V)$ and $B^i_g \in \mathbf{S}_{p-t}(V)$; thus (7) gives

$$\binom{n+p-s-t}{n-s}\langle A^i_f \cdot D^j_g, C^j_f \cdot B^i_g\rangle$$

$$= \sum_{w=0}^{\mu_{s,t}}\binom{n-s}{w}\binom{p-t}{w}\langle A^i_f \mathbin{\bar{\otimes}}_w B^i_g, C^j_f \mathbin{\bar{\otimes}}_w D^j_g\rangle$$

$$= \sum_{w=0}^{\mu_{s,t}}\binom{n-s}{w}\binom{p-t}{w}\langle \mathbf{C}_w(A^i_f \mathbin{\bar{\otimes}} B^i_g), \mathbf{C}_w(C^j_f \mathbin{\bar{\otimes}} D^j_g)\rangle, \tag{16}$$

where $\mu_{s,t} = \min\{n-s, p-t\}$. Contraction and insertion are commuting operations, so we have

$$\sum_{f\in\Gamma_s}\sum_{g\in\Gamma_t}\langle \mathbf{C}_w(A^i_f \mathbin{\bar{\otimes}} B^i_g), \mathbf{C}_w(C^j_f \mathbin{\bar{\otimes}} D^j_g)\rangle = \langle \mathbf{C}_w(A^i \mathbin{\bar{\otimes}} B^i), \mathbf{C}_w(C^j \mathbin{\bar{\otimes}} D^j)\rangle \tag{17}$$

for each i and j. Thus, summing (16) over i, j, f, and g, and noting (17), we obtain

$$
\begin{aligned}
&\binom{n+p-s-t}{n-s} \sum_{i=1}^{u}\sum_{j=1}^{v}\sum_{f\in\Gamma_s}\sum_{g\in\Gamma_t}\langle A_f^i\cdot D_g^j, C_f^j\cdot B_g^i\rangle \\
&\quad=\sum_{i,j}\sum_{f\in\Gamma_s}\sum_{g\in\Gamma_t}\sum_{w=0}^{\mu_{s,t}}\binom{n-s}{w}\binom{p-t}{w}\langle \mathbf{C}_w(A_f^i\mathbin{\bar\otimes} B_g^i), \mathbf{C}_w(C_f^j\mathbin{\bar\otimes} D_g^j)\rangle \\
&\quad=\sum_{i,j}\sum_{w=0}^{\mu_{s,t}}\binom{n-s}{w}\binom{p-t}{w}\left\{\sum_{f\in\Gamma_s}\sum_{g\in\Gamma_t}\langle \mathbf{C}_w(A_f^i\mathbin{\bar\otimes} B_g^i), \mathbf{C}_w(C_f^j\mathbin{\bar\otimes} D_g^j)\rangle\right\} \\
&\quad=\sum_{i,j}\sum_{w=0}^{\mu_{s,t}}\binom{n-s}{w}\binom{p-t}{w}\langle \mathbf{C}_w(A^i\mathbin{\bar\otimes} B^i), \mathbf{C}_w(C^j\mathbin{\bar\otimes} D^j)\rangle \\
&\quad=\sum_{w=0}^{\mu_{s,t}}\binom{n-s}{w}\binom{p-t}{w}\left\langle \mathbf{C}_w\left(\sum_i A^i\mathbin{\bar\otimes} B^i\right), \mathbf{C}_w\left(\sum_j C^j\mathbin{\bar\otimes} D^j\right)\right\rangle \\
&\quad=\sum_{w=0}^{\mu_{s,t}}\binom{n-s}{w}\binom{p-t}{w}\langle \mathbf{C}_w(F), \mathbf{C}_w(G)\rangle, \qquad (18)
\end{aligned}
$$

which is the desired identity. □

Theorem 2, in conjunction with (13), implies that

$$[F,G]_{s,t}=\sum_{w=0}^{\mu_{s,t}}\binom{n-s}{w}\binom{p-t}{w}\langle \mathbf{C}_w(F), \mathbf{C}_w(G)\rangle \quad \forall F,G\in \mathbf{S}_{n,p}(V). \qquad (19)$$

Since (19) does not refer to expansions, $\sum_i A^i \mathbin{\bar\otimes} B^i$ and $\sum_j C^j \mathbin{\bar\otimes} D^j$, that represent F and G, respectively, $[\cdot,\cdot]_{s,t}$ is well-defined for each s and t. Of course (19) could have been given as the definition of $[F,G]_{s,t}$, but Theorem 2 would still be required to prove our main result, as it is (12) that provides the most useful expression for $[F,G]_{s,t}$.

Substituting $G=F$ in (19), and recalling that $\mathbf{C}_0$ is the identity map, we obtain

$$[F,F]_{s,t}=\sum_{w=0}^{\mu_{s,t}}\binom{n-s}{w}\binom{p-t}{w}\|\mathbf{C}_w(F)\|^2\geqslant\|\mathbf{C}_0(F)\|^2=\|F\|^2; \qquad (20)$$

thus, $[\cdot,\cdot]_{s,t}$ is also positive definite for each s and t.

Note that $\mu_{s,t}\geqslant\mu_{s+1,t}$ and $\binom{n-s}{\nu}\geqslant\binom{n-s-1}{\nu}$ for each s, $0\leqslant s\leqslant\kappa-1$; thus

$$
\begin{aligned}
[F,F]_{s,t}-[F,F]_{s+1,t}&=\sum_{w=0}^{\mu_{s,t}}\binom{n-s}{w}\binom{p-t}{w}\|\mathbf{C}_w(F)\|^2 \\
&\quad-\sum_{w=0}^{\mu_{s+1,t}}\binom{n-s-1}{w}\binom{p-t}{w}\|\mathbf{C}_w(F)\|^2 \\
&\geqslant\sum_{w=0}^{\mu_{s+1,t}}\binom{p-t}{w}\left(\binom{n-s}{w}-\binom{n-s-1}{w}\right)\|\mathbf{C}_w(F)\|^2 \\
&\geqslant 0 \qquad (21)
\end{aligned}
$$

for all $F \in \mathbf{S}_{n,p}(V)$. In a similar manner we can show that $[F, F]_{s,t} \geqslant [F, F]_{s,t+1}$ for all $F \in \mathbf{S}_{n,p}(V)$ provided that $0 \leqslant s \leqslant \kappa$ and $0 \leqslant t < \kappa$. We have proved our main result.

Theorem 3. *If s and t are integers such that $0 \leqslant s \leqslant \kappa$ and $0 \leqslant t \leqslant \kappa$, then $[\cdot, \cdot]_{s,t}$ is a positive definite sesquilinear form on $\mathbf{S}_{n,p}(V)$, and hence an inner product on $\mathbf{S}_{n,p}(V)$. Moreover, if u and v are integers such that $0 \leqslant s \leqslant u \leqslant \kappa$ and $0 \leqslant t \leqslant v \leqslant \kappa$, then $[F, F]_{s,t} \geqslant [F, F]_{u,v}$ for all $F \in \mathbf{S}_{n,p}(V)$.*

Our next theorem is an extension of Theorem 4 of [14] to the non-decomposable case. If $F \in \mathbf{S}_{n,p}(V)$, then insertions such as $F(x_1, x_2, \ldots, x_r; y_1, y_2, \ldots, y_s)$, where $r \leqslant n$ and $s \leqslant p$, were defined previously. Note that if $F = A \bar{\otimes} B$ and $x \in V$, where $A \in \mathbf{S}_n(V)$ and $B \in \mathbf{S}_p(V)$, then $F(x; \cdot) = A(x) \bar{\otimes} B$, and $F(\cdot; x) = A \bar{\otimes} B(x)$. When it is necessary to indicate the dependence of $[\cdot, \cdot]_{s,t}$ on n and p we will write $[\cdot, \cdot]_{n,p,s,t}$ instead of $[\cdot, \cdot]_{s,t}$, and $\|\cdot\|_{n,p,s,t}$ instead of $\|\cdot\|_{s,t}$. We require the following.

Lemma 1. *If $F \in \mathbf{S}_{n,p}(V)$, $0 \leqslant s \leqslant \kappa$, and $0 \leqslant t \leqslant \kappa$, then $\|F\|^2_{n,p,s+1,t} = \sum_{r=1}^m \|F(e_r; \cdot)\|^2_{n-1,p,s,t}$ when $s \leqslant \kappa - 1$, and $\|F\|^2_{n,p,s,t+1} = \sum_{r=1}^m \|F(\cdot; e_r)\|^2_{n,p-1,s,t}$ when $t \leqslant \kappa - 1$,*

Proof. Note that $(\mathbf{C}_\nu(F))(x; \cdot) = \mathbf{C}_\nu(F(x; \cdot))$ for any $x \in V$; moreover, for any $H \in \mathbf{S}_{n,p}(V)$, we have $\|H\|^2 = \sum_{r=1}^m \|H(e_r; \cdot)\|^2$. Therefore,

$$\|\mathbf{C}_\nu(F)\|^2 = \sum_{r=1}^m \|(\mathbf{C}_\nu(F))(e_r; \cdot)\|^2 = \sum_{r=1}^m \|\mathbf{C}_\nu(F(e_r; \cdot))\|^2;$$

so, letting α denote $\min\{n - s - 1, p - t\}$, and invoking (19) we obtain that

$$\begin{aligned}\|F\|^2_{n,p,s+1,t} &= \sum_{\nu=0}^{\alpha} \binom{n-s-1}{\nu}\binom{p-t}{\nu} \|\mathbf{C}_\nu(F)\|^2 \\ &= \sum_{r=1}^m \left\{\sum_{\nu=0}^{\alpha} \binom{n-s-1}{\nu}\binom{p-t}{\nu} \|\mathbf{C}_\nu(F(e_r; \cdot))\|^2\right\}. \end{aligned} \tag{22}$$

We have $F(e_r; \cdot) \in \mathbf{S}_{n-1,p}(V)$ for each r, $1 \leqslant r \leqslant m$; thus, invoking (19) once again we obtain

$$\|F(e_r; \cdot)\|^2_{n-1,p,s,t} = \sum_{\nu=1}^{\beta} \binom{n-1-s}{\nu}\binom{p-t}{\nu} \|\mathbf{C}_\nu(F(e_r; \cdot))\|^2, \tag{23}$$

where $\beta = \min\{(n-1) - s, p - t\}$. Since $\alpha = \beta$ the right side of (23) is identical to the summand in (22). We therefore substitute to obtain

$$\|F\|^2_{n,p,s+1,t} = \sum_{r=1}^m \|F(e_r; \cdot)\|^2_{n-1,p,s,t}$$

as required. The second identity is proved similarly. $\square$

Theorem 4. *If $F \in \mathbf{S}_{n,p}(V)$, then*

$$\|F\|^2_{n,p,s,t} \geqslant \sum_{r=1}^m \|F(e_r; \cdot)\|^2_{n-1,p,s,t} \quad \textit{and} \quad \|F\|^2_{n,p,s,t} \geqslant \sum_{r=1}^m \|F(\cdot; e_r)\|^2_{n,p-1,s,t}.$$

Proof. Theorem 3 and Lemma 1 imply that

$$\|F\|^2_{n,p,s,t} \geqslant \|F\|^2_{n,p,s+1,t} = \sum_{r=1}^{m} \|F(e_r;\cdot)\|^2_{n-1,p,s,t}.$$

The second inequality it true because $\|F\|^2_{n,p,s,t} \geqslant \|F\|^2_{n,p,s,t+1} = \sum_{r=1}^{m} \|F(\cdot;e_r)\|^2_{n,p-1,s,t}$. $\square$

3. Inequalities for norms of symmetric multilinear functions

It is surprising how many known inequalities follow from Theorem 3 by specializing to the case $F = A \bar{\otimes} B$. In this section we list several such inequalities as corollaries, and present a few new results. The first of our corollaries is Neuberger's inequality (9). Permanental inequalities are obtained by specializing Theorem 3, or one of its corollaries, to the case when F is fully decomposable. We present such results in Section 4.

Corollary 1. *If $A \in \mathbf{S}_n(V)$ and $B \in \mathbf{S}_p(V)$, then $\binom{n+p}{n}\|A \cdot B\|^2 \geqslant \|A\|^2\|B\|^2$.*

Proof. Since $\mu_{0,0} = \kappa$ and $\mu_{\kappa,\kappa} = 0$, (14) and (19) imply that if $A \in \mathbf{S}_n(V)$ and $B \in \mathbf{S}_p(V)$, then

$$[A \bar{\otimes} B, A \bar{\otimes} B]_{0,0} = \binom{n+p}{n} \|A \cdot B\|^2 \quad \text{and} \quad [A \bar{\otimes} B, A \bar{\otimes} B]_{\kappa,\kappa} = \|A\|^2\|B\|^2. \tag{24}$$

But, according to Theorem 3, we have $[F,F]_{0,0} \geq [F,F]_{\kappa,\kappa}$ for all $F \in \mathbf{S}_{n,p}(V)$; thus, setting $F = A \bar{\otimes} B$, we obtain that $[A \bar{\otimes} B, A \bar{\otimes} B]_{0,0} \geqslant [A \bar{\otimes} B, A \bar{\otimes} B]_{\kappa,\kappa}$, which, on account of (24), is what we wished to prove. $\square$

If $1 \leqslant i \leqslant m$, and $D \in \mathbf{S}_q(V)$ for some q, then D_i is the insertion of e_i into D; thus, $D_i(x_1, x_2, \ldots, x_{q-1}) = D(e_i, x_1, x_2, \ldots, x_{q-1})$ for all $x_1, x_2, \ldots, x_{q-1} \in V$. The following is Theorem 4 of [14].

Corollary 2. *If $A \in \mathbf{S}_n(V)$ and $B \in \mathbf{S}_p(V)$, then*

$$\|A \cdot B\|^2 \geqslant \frac{n}{n+p} \sum_{i=1}^{m} \|A_i \cdot B\|^2 \quad \textit{and} \quad \|A \cdot B\|^2 \geqslant \frac{p}{n+p} \sum_{i=1}^{m} \|A \cdot B_i\|^2. \tag{25}$$

Proof. Let $F = A \bar{\otimes} B$. As noted above, $[F,F]_{0,0} = \binom{n+p}{n}\|A \cdot B\|^2$; moreover, (12) indicates that

$$[F,F]_{1,0} = \binom{n+p-1}{n-1} \sum_{i=1}^{m} \|A_i \cdot B\|^2. \tag{26}$$

But, according to Theorem 3, we have $[F,F]_{0,0} \geqslant [F,F]_{1,0}$; therefore we have

$$\binom{n+p}{n} \|A \cdot B\|^2 \geqslant \binom{n+p-1}{n-1} \sum_{i=1}^{m} \|A_i \cdot B\|^2, \tag{27}$$

which is equivalent to the first inequality in (25). The second inequality arises by setting $F = A \bar{\otimes} B$ in the inequality $[F, F]_{0,0} \geqslant [F, F]_{0,1}$. $\square$

That (25) refines Neuberger's inequality also follows from Theorem 3 because it says that $[F, F]_{0,0} \geqslant [F, F]_{1,0} \geqslant [F, F]_{\kappa,\kappa}$ for all $F \in \mathbf{S}_{n,p}(V)$. This, along with (24) and (26), imply that if $F = A \bar{\otimes} B$, then

$$\|A \cdot B\|^2 \geqslant \frac{n}{n+p} \sum_{i=1}^{m} \|A_i \cdot B\|^2 \geqslant \left[\frac{n!p!}{(n+p)!}\right] \|A\|^2 \|B\|^2.$$

To specialize Theorem 3 to the case $F = A \bar{\otimes} B$ we define functions $\Lambda_{s,t}$ in accordance with [14]. Thus,

$$\Lambda_{s,t}(A, B) = \left[\frac{n_{(s)} p_{(t)}}{(n+p)_{(s+t)}}\right] \sum_{f \in \Gamma_s} \sum_{g \in \Gamma_t} \|A_f \cdot B_g\|^2 \quad \forall A \in \mathbf{S}_n(V)\ \forall B \in \mathbf{S}_p(V), \tag{28}$$

where the standard factorial $x_{(t)}$ is $x(x-1)(x-2)\cdots(x-t+1)$ when $1 \leqslant t \leqslant x$, and 1 when $t = 0$. Thus, $\Lambda_{0,0}(A, B) = \|A \cdot B\|^2$, $\Lambda_{1,0}(A, B) = [n/(n+p)] \sum_{i=1}^{m} \|A_i \cdot B\|^2$, and $\Lambda_{\kappa,\kappa}(A, B) = [1/\binom{n+p}{n}] \|A\|^2 \|B\|^2$. Since

$$\binom{n+p-s-t}{n-s} = \binom{n+p}{n} \left[\frac{n_{(s)} p_{(t)}}{(n+p)_{(s+t)}}\right]$$

(12) and (28) imply that if $F = A \bar{\otimes} B$, then $\binom{n+p}{n}[F, F]_{s,t} = \Lambda_{s,t}(A, B)$. Therefore, the below, which is Theorem 5 of [14], is an immediate consequence of Theorem 3.

Corollary 3. *Suppose* $0 \leqslant s \leqslant u \leqslant \kappa$, *and* $0 \leqslant t \leqslant v \leqslant \kappa$. *Then*, $\Lambda_{s,t}(A, B) \geqslant \Lambda_{u,v}(A, B)$ *for all* $A \in \mathbf{S}_n(V)$ *and* $B \in \mathbf{S}_p(V)$; *that is*,

$$\left[\frac{n_{(s)} p_{(t)}}{(n+p)_{(s+t)}}\right] \sum_{f \in \Gamma_s} \sum_{g \in \Gamma_t} \|A_f \cdot B_g\|^2 \geqslant \left[\frac{n_{(u)} p_{(v)}}{(n+p)_{(u+v)}}\right] \sum_{f \in \Gamma_u} \sum_{g \in \Gamma_v} \|A_f \cdot B_g\|^2. \tag{29}$$

For example, the inequality $\Lambda_{0,0}(A, B) \geqslant \Lambda_{1,0}(A, B) \geqslant \Lambda_{1,1}(A, B) \geqslant \Lambda_{\kappa,\kappa}(A, B)$ transforms into

$$\begin{aligned}\|A \cdot B\|^2 &\geqslant \frac{n}{n+p} \sum_{i=1}^{m} \|A_i \cdot B\|^2 \geqslant \frac{np}{(n+p)(n+p-1)} \sum_{i=1}^{m} \sum_{j=1}^{m} \|A_i \cdot B_j\|^2 \\ &\geqslant \frac{n!p!}{(n+p)!} \|A\|^2 \|B\|^2.\end{aligned} \tag{30}$$

Our next corollary is Theorem 1 of [6]. If $f \in V^*$, and k is a positive integer, then by f^k we mean the member of $\mathbf{S}_k(V)$ defined by $f^k(x_1, x_2, \ldots, x_k) = \prod_{i=1}^{k} f(x_i)$ for all $x_1, x_2, \ldots, x_k \in V$.

Corollary 4. *If* $f \in V^*$, *and* $B \in \mathbf{S}_p(V)$, *then* $\|f^n \cdot B\|^2 \geqslant [n/(n+p)] \|f\|^2 \|f^{n-1} \cdot B\|^2$.

Proof. It is clear that $f^n(e_i) = f(e_i) f^{n-1}$, and we know that $\|f\|^2 = \sum_{i=1}^{m} |f(e_i)|^2$; hence,

$$\sum_{i=1}^{m} \|f^n(e_i) \cdot B\|^2 = \sum_{i=1}^{m} |f(e_i)|^2 \|f^{n-1} \cdot B\|^2 = \|f\|^2 \|f^{n-1} \cdot B\|^2. \tag{31}$$

But, Corollary 2, indicates that $\|f^n \cdot B\|^2 \geqslant [n/(n+p)] \sum_{i=1}^m \|f^n(e_i) \cdot B\|^2$. Combining this with the result of multiplying (31) by $[n/(n+p)]$ we obtain that $\|f^n \cdot B\|^2 \geqslant [n/(n+p)]\|f\|^2 \|f^{n-1} \cdot B\|^2$ as required. □

Interesting results also arise by applying the Cauchy-Schwartz inequality to the inner products $[\cdot,\cdot]_{s,t}$. Two such results appear below. The first of these is the second theorem and main result of [11]. The second inequality in Corollary 6 is contained in Theorem 3 of [11]; the first inequality is new.

Corollary 5. *If $A, C \in \mathbf{S}_n(V)$ and $B, D \in \mathbf{S}_p(V)$, then $|\langle A \cdot B, C \cdot D\rangle|^2 \leqslant \|A \cdot D\|^2 \|C \cdot B\|^2$.*

Proof. Let $k = \binom{n+p}{n}$. Then, (14) says that $[A \bar{\otimes} D, C \bar{\otimes} B]_{0,0} = k\langle A \cdot B, C \cdot D\rangle$. Similarly, $[A\bar{\otimes}D, A\bar{\otimes}D]_{0,0} = k\langle A \cdot D, A \cdot D\rangle = k\|A \cdot D\|^2$, and $[C\bar{\otimes}B, C \bar{\otimes} B]_{0,0} = k\langle C \cdot B, C \cdot B\rangle = k\|C \cdot B\|^2$. Applying Cauchy-Schwartz to $[\cdot,\cdot]_{0,0}$ we determine that

$$
\begin{aligned}
k^2|\langle A \cdot B, C \cdot D\rangle|^2 &= |[A \bar{\otimes} D, C \bar{\otimes} B]_{0,0}|^2 \leqslant [A \bar{\otimes} D, A \bar{\otimes} D]_{0,0}[C \bar{\otimes} B, C \bar{\otimes} B]_{0,0} \\
&= k^2\langle A \cdot D, A \cdot D\rangle\langle C \cdot B, C \cdot B\rangle = k^2\|A \cdot D\|^2\|C \cdot B\|^2, \qquad (32)
\end{aligned}
$$

which immediately implies that $|\langle A \cdot B, C \cdot D\rangle|^2 \leqslant \|A \cdot D\|^2\|C \cdot B\|^2$, as required. □

Corollary 6. *If $E, F \in \mathbf{S}_n(V)$ and $G \in \mathbf{S}_p(V)$, then*

$$
|\langle E \cdot E \cdot G, F \cdot F \cdot G\rangle|^2 \leqslant \|E \cdot F \cdot G\|^4 \leqslant \|E \cdot E \cdot G\|^2\|F \cdot F \cdot G\|^2. \qquad (33)
$$

Proof. Since $F \cdot E \cdot G = E \cdot F \cdot G$, Corollary 5 with $A = E$, $B = F \cdot G$, $C = F$, and $D = E \cdot G$, says that

$$
\|E \cdot F \cdot G\|^4 \leqslant \|E \cdot E \cdot G\|^2\|F \cdot F \cdot G\|^2. \qquad (34)
$$

Applying Corollary 5 with $A = E$, $B = E \cdot G$, $C = F$, and $D = F \cdot G$ we obtain

$$
|\langle E \cdot E \cdot G, F \cdot F \cdot G\rangle|^2 \leqslant \|E \cdot F \cdot G\|^2\|F \cdot E \cdot G\|^2 = \|E \cdot F \cdot G\|^4. \qquad (35)
$$

Combining (34) and (35) we obtain (33). □

4. Inequalities involving permanents of gram matrices

A gram matrix is a positive semidefinite Hermitian matrix. Theorem 3 and its corollaries in the previous section yield some interesting inequalities involving permanents of such matrices. Let $\mathscr{H}_q$ denote the set of all $q \times q$ positive semidefinite Hermitian matrices. Given any complex vector space W with inner product $\langle\langle\cdot,\cdot\rangle\rangle$ and vectors $x_1, x_2, \ldots, x_q, y_1, y_2, \ldots, y_q$ in W, we let $\mathrm{Gr}(x_1, x_2, \ldots, x_q; y_1, y_2, \ldots, y_q)$ denote the $q \times q$ matrix, whose (i, j)th entry is $\langle\langle x_i, y_j\rangle\rangle$ for each i and j. If $y_i = x_i$ for each i, then $\mathrm{Gr}(x_1, x_2, \ldots, x_q; y_1, y_2, \ldots, y_q)$, denoted by $\mathrm{Gr}(x_1, x_2, \ldots, x_q)$, is the Gram matrix generated by $x_1, x_2, \ldots, x_q$. As is well known, every $q \times q$ Gram matrix, $\mathrm{Gr}(x_1, x_2, \ldots, x_q)$, is in $\mathscr{H}_q$; moreover, if the dimension of W is at least q, then the set of all Gram matrices $\mathrm{Gr}(x_1, x_2, \ldots, x_q)$ is precisely $\mathscr{H}_q$. This follows from the Cholesky factorization, or by the extension of [1, Theorem 7.2.7] to the positive semi-definite case: see [1, Problem 9, page 409].

We shall apply Theorem 3 and corollaries to symmetric multilinear functions obtained by taking symmetric products of members of $V^* = \mathbf{T}_1(V)$. Note that if $x, y \in V$, then $\langle x, y\rangle = \langle \tilde{y}, \tilde{x}\rangle$; thus, $\mathrm{Gr}(x_1, x_2, \ldots, x_q)$ is the transpose of $\mathrm{Gr}(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_q)$ for all $x_1, x_2, \ldots, x_q \in V$. Therefore, $\mathscr{H}_q$ is also the set of all Gram matrices of the form $\mathrm{Gr}(f_1, f_2, \ldots, f_q)$ where $f_1, f_2, \ldots, f_q \in V^*$. The key fact that allows us to obtain permanental inequalities from Theorem 3 is that if $f_1, f_2, \ldots, f_q, g_1, g_2, \ldots, g_q \in V^*$, then

$$q!\langle f_1 \cdot f_2 \cdots f_q, g_1 \cdot g_2 \cdot \ldots \cdot g_q\rangle = \mathrm{per}(\mathrm{Gr}(f_1, f_2, \ldots, f_q; g_1, g_2, \ldots, g_q)). \tag{36}$$

In particular, if $x_1, x_2, \ldots, x_q \in V$, then, as noted previously in [3, Theorem 2.2], we have

$$q!\|\tilde{x}_1 \cdot \tilde{x}_2 \cdots \tilde{x}_q\|^2 = \mathrm{per}(\mathrm{Gr}(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_q)) = \mathrm{per}(\mathrm{Gr}(x_1, x_2, \ldots, x_q)). \tag{37}$$

Lieb's permanental inequality, presented below, is included among the results implied by Theorem 3.

Corollary 7. *If $M \in \mathscr{H}_{n+p}$ and M is partitioned in the form $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{bmatrix}$ where M_{11} is $n \times n$ and M_{22} is $p \times p$, then* $\mathrm{per}(M) \geqslant \mathrm{per}(M_{11})\mathrm{per}(M_{22})$ *with equality if and only if M has a row or column of zeros or M_{12} is the zero matrix.*

Proof. Choose $f_1, f_2, \ldots, f_{n+p} \in V^*$ such that $M = \mathrm{Gr}(f_1, f_2, \ldots, f_{n+p})$. Then $M_{11} = \mathrm{Gr}(f_1, f_2, \ldots, f_n)$, and $M_{22} = \mathrm{Gr}(f_{n+1}, f_{n+2}, \ldots, f_{n+p})$. Let $A = f_1 \cdot f_2 \cdot \ldots \cdot f_n$, and let $B = f_{n+1} \cdot f_{n+2} \cdot \ldots \cdot f_{n+p}$. Then,

$$\begin{aligned}\mathrm{per}(M) &= \mathrm{per}(\mathrm{Gr}(f_1, f_2, \ldots, f_{n+p})) = (n+p)!\|A \cdot B\|^2 \geq [n!p!]\|A\|^2\|B\|^2 \\ &= \mathrm{per}(\mathrm{Gr}(f_1, f_2, \ldots, f_n))\mathrm{per}(\mathrm{Gr}(f_{n+1}, f_{n+2}, \ldots, f_{n+p})) \\ &= \mathrm{per}(M_{11})\mathrm{per}(M_{22}).\end{aligned} \tag{38}$$

This proves the inequality. If M has a row or column of zeros, or if M_{12} is the zero matrix, then it is obvious that $\mathrm{per}(M) = \mathrm{per}(M_{11})\mathrm{per}(M_{22})$ since in the former case both sides reduce to 0, and in the latter case M is block diagonal. □

Lemma 2. *Suppose $H \in \mathbf{S}_k(V)$. If $x, y \in V$, then*

$$(\tilde{y} \cdot H)(x) = [1/(k+1)]\langle x, y\rangle H + [k/(k+1)]\tilde{y} \cdot H(x). \tag{39}$$

Moreover, if $f_1, f_2, \ldots, f_k \in V^*$, then

$$(f_1 \cdot f_2 \cdots f_k)(x) = [1/k]\sum_{i=1}^{k} f_i(x) f_1 \cdot f_2 \cdots f_{i-1} \cdot f_{i+1} \cdots f_k, \tag{40}$$

and if $y_1, y_2, \ldots, y_k \in V$, then

$$(\tilde{y}_1 \cdot \tilde{y}_2 \cdots \tilde{y}_k)(x) = [1/k]\sum_{i=1}^{k}\langle x, y_i\rangle \tilde{y}_1 \cdot \tilde{y}_2 \cdots \tilde{y}_{i-1} \cdot \tilde{y}_{i+1} \cdots \tilde{y}_k. \tag{41}$$

Proof. Eq. (39) is the second part of Lemma 3 of [14]; moreover, (41) follows immediately from (40). We prove (40) by induction. If $k = 1$, then (40) is obviously true. Suppose (40) is true for $k-1$ where $k \geqslant 2$. There exists $y \in V$ such that $\tilde{y} = f_k$. Let $H = f_1 \cdot f_2 \cdots f_{k-1}$. Then, applying (39) to $f_k \cdot H$ we obtain

$$(\tilde{y}_1 \cdot \tilde{y}_2 \cdots \tilde{y}_k)(x) = [1/k]\langle x, y\rangle H + [(k-1)/k]\tilde{y} \cdot H(x)$$

$$= [1/k]f_k(x)H + [(k-1)/k]f_k \cdot H(x) \tag{42}$$

Now $H = f_1 \cdot f_2 \cdots f_{k-1} \in \mathbf{S}_{k-1}(V)$, so we employ the inductive hypothesis to obtain that

$$H(x) = [1/(k-1)]\sum_{i=1}^{k-1} f_i(x) f_1 \cdot f_2 \cdots f_{i-1} \cdot f_{i+1} \cdots f_{k-1}. \tag{43}$$

Substituting (43) into (42) we obtain (41). □

If $s \leqslant t$, then let $Q_{s,t}$ denote the set of all increasing functions from $\{1, 2, \ldots, s\}$ to $\{1, 2, \ldots, t\}$. Given a $t \times t$ matrix M and $\alpha, \beta \in Q_{s,t}$ we let $M[\alpha|\beta]$ denote the $s \times s$ submatrix of M comprised of the rows indicated by α, and the columns indicated by β; that is, $M[\alpha|\beta] = [m_{i,j}]$ where $m_{i,j} = M_{\alpha(i),\beta(j)}$ for each i and j, $1 \leqslant i, j \leqslant s$. By $M(\alpha|\beta)$ we shall mean the $(t-s) \times (t-s)$ submatrix of M that is complementary to $M[\alpha|\beta]$. For example, in the following, which is is still another corollary to Theorem 3, we refer to $G(s|t)$ where G is an $n \times n$ matrix; in this case $G(s|t)$ denotes the $(n-1) \times (n-1)$ submatrix obtained from G by deleting row s and column t. The below is also special case of Theorem 1 of [7].

Corollary 8. *Suppose* $1 \leqslant k \leqslant n$. *If* $G = [g_{ij}] \in \mathscr{H}_n$, *then* $\operatorname{per}(G) \geqslant [1/k]\sum_{i,j=1}^{k} g_{ij}\operatorname{per}(G(i|j))$.

Proof. Note that if $k = n$, then the inequality of the lemma reduces to equality; so we assume that $k < n$. Let $f_1, f_2, \ldots, f_n$ be members of V^* such that $G = \mathrm{Gr}(f_1, f_2, \ldots, f_n)$. Let $A = f_1 \cdot f_2 \cdots f_k$ and let $B = f_{k+1} \cdot f_{k+2} \cdots f_n$. For each s, $1 \leqslant s \leqslant k$, let $A^{(s)}$ denote $f_1 \cdot f_2 \cdots f_{s-1} \cdot f_{s+1} \cdots f_k$. By (41) we have $A_i = [1/k]\sum_{s=1}^{k} f_s(e_i)A^{(s)}$ for each i, $1 \leqslant i \leqslant k$. If s and t satisfy $1 \leqslant s, t \leqslant k$, then we have $(n-1)!\langle A^{(s)} \cdot B, A^{(t)} \cdot B\rangle = \operatorname{per}(G(s|t))$ by (37); thus, by (37) and Corollary 3 we have

$$\begin{aligned}
\operatorname{per}(G) &= \operatorname{per}(\mathrm{Gr}(f_1, f_2, \ldots, f_n)) = n!\|A \cdot B\|^2 \geqslant n![k/n]\sum_{i=1}^{m} \|A_i \cdot B\|^2 \\
&= [1/k]\sum_{i=1}^{m}\sum_{s=1}^{k}\sum_{t=1}^{k} f_s(e_i)\overline{f_t(e_i)}\{(n-1)!\langle A^{(s)} \cdot B, A^{(t)} \cdot B\rangle\} \\
&= [1/k]\sum_{s=1}^{k}\sum_{t=1}^{k}\left\{\sum_{i=1}^{m} f_s(e_i)\overline{f_t(e_i)}\right\}\operatorname{per}(G(s|t)) \\
&= [1/k]\sum_{s=1}^{k}\sum_{t=1}^{k}\langle f_s, f_t\rangle\operatorname{per}(G(s|t)) = [1/k]\sum_{s=1}^{k}\sum_{t=1}^{k} g_{st}\operatorname{per}(G(s|t)),
\end{aligned} \tag{44}$$

which is the stated inequality. If G is the $n \times n$ identity matrix, then our inequality reduces to equality; thus it is sharp and cannot be improved without placing additional restrictions on G. □

Corollary 6 also provides an interesting permanental inequality. We make the following definition to simplify its statement: if $u, \hat{u}, v, \hat{v}, w$ and $\hat{w}$ are sequences of members of V of lengths r, r, s, s, t and t, respectively, then we let $\mathrm{Gr}(u, v, w; \hat{u}, \hat{v}, \hat{w})$ denote the mixed Gram matrix

$$\mathrm{Gr}(u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_s, w_1, \dots, w_t; \hat{u}_1, \hat{u}_2, \dots, \hat{u}_r, \hat{v}_1, \hat{v}_2, \dots, \hat{v}_s, \hat{w}_1, \dots, \hat{w}_t).$$

If $u = \hat{u}$, $v = \hat{v}$, and $w = \hat{w}$, then we will write $\mathrm{Gr}(u, v, w)$. The first inequality appearing in Corollary 9 is new; the second originally appeared in [11].

Corollary 9. *If $x = \{x_i\}_{i=1}^n$, $y = \{y_j\}_{j=1}^n$, and $z = \{z_k\}_{k=1}^p$ are sequences of members of V, then*

$$\begin{aligned} |\mathrm{per}(\mathrm{Gr}(x, x, z; y, y, z))| &\leqslant \mathrm{per}(\mathrm{Gr}(x, y, z)) \\ &\leqslant \sqrt{\mathrm{per}(\mathrm{Gr}(x, x, z))}\sqrt{\mathrm{per}(\mathrm{Gr}(y, y, z))}. \end{aligned} \tag{45}$$

Proof. Let $A = \tilde{x}_1 \cdot \tilde{x}_2 \cdots \tilde{x}_n$, let $B = \tilde{y}_1 \cdot \tilde{y}_2 \cdots \tilde{y}_n$, and let $C = \tilde{z}_1 \cdot \tilde{z}_2 \cdots \tilde{z}_p$. Then by (37)

$$(2n + p)!\langle A \cdot A \cdot C, B \cdot B \cdot C\rangle = \mathrm{per}(\mathrm{Gr}(x, x, z; y, y, z)). \tag{46}$$

Similarly, we have $(2n + p)!\|A \cdot B \cdot C\|^2 = \mathrm{per}(\mathrm{Gr}(x, y, z))$, $\sqrt{(2n + p)!}\|A \cdot A \cdot C\| = \sqrt{\mathrm{per}(\mathrm{Gr}(x, x, z))}$, and $\sqrt{(2n + p)!}\|B \cdot B \cdot C\| = \sqrt{\mathrm{per}(\mathrm{Gr}(y, y, z))}$. By Corollary 6,

$$|\langle A \cdot A \cdot C, B \cdot B \cdot C\rangle| \leqslant \|A \cdot B \cdot C\|^2 \leqslant \|A \cdot A \cdot C\|\|B \cdot B \cdot C\|. \tag{47}$$

Multiplication of (47) by $(2n + p)!$ followed by substitution of the permanental equivalents of the various terms immediately produces (45). $\square$

The following results from specializing Corollary 4 to the decomposable case.

Corollary 10. *If $M = [m_{ij}] \in \mathscr{H}_{n+p}$ and the $n \times n$ submatrix $M[1, 2, \dots, n|1, 2, \dots, n] = [m_{ij}]_{i,j=1}^n$ has rank one, then $\mathrm{per}(M) \geqslant nm_{ii}\mathrm{per}(M(i|i))$ for each i, $1 \leqslant i \leqslant n$.*

Proof. It is sufficient to consider the case $i = 1$; that is, we will prove that $\mathrm{per}(M) \geqslant nm_{11}\mathrm{per}(M(1|1))$. Let $f_1, f_2, \dots, f_{n+p}$ be members of V^* such that $\mathrm{Gr}(f_1, f_2, \dots, f_{n+p}) = M$. Note that if $\alpha \in \mathbb{C}$, then

$$\mathrm{per}(\mathrm{Gr}(g_1, g_2, \dots, g_{i-1}, \alpha g_i, g_{i+1}, g_{i+2}, \dots, g_{n+p})) = |\alpha|^2\mathrm{per}(\mathrm{Gr}(g_1, g_2, \dots, g_{n+p}))$$

for all $g_1, g_2, \dots, g_{n+p} \in V^*$. Since the rank of $\mathrm{Gr}(f_1, f_2, \dots, f_n)$ is one, if we let $f = [1/\|f_1\|]f_1$, then for each i, $1 \leqslant i \leqslant n$, there exists a non-zero number α_i such that $f_i = \alpha_i f$. Let $B = f_{n+1} \cdot f_{n+2} \cdots f_{n+p}$. By Corollary 4 we have $\|f^n \cdot B\|^2 \geqslant [n/(n + p)]\|f\|^2\|f^{n-1} \cdot B\|^2$. Therefore, letting $f^{\{q\}}$ abbreviate the vector sequence $f, f, \dots, f$-(q copies), we obtain

$$\begin{aligned} \mathrm{per}(M) &= \mathrm{per}(\mathrm{Gr}(f_1, f_2, \dots, f_{n+p})) \\ &= \mathrm{per}(\mathrm{Gr}(\alpha_1 f, \alpha_2 f, \dots, \alpha_n f, f_{n+1}, f_{n+2}, \dots, f_{n+p})) \\ &= \left\{\prod_{i=1}^n |\alpha_i|^2\right\} \mathrm{per}(\mathrm{Gr}(f^{\{n\}}, f_{n+1}, f_{n+2}, \dots, f_{n+p})) \\ &= \left\{\prod_{i=1}^n |\alpha_i|^2 (n + p)!\right\} \|f^n \cdot B\|^2 \\ &\geqslant \left\{\prod_{i=1}^n |\alpha_i|^2 (n + p)!\right\} [n/(n + p)]\|f\|^2\|f^{n-1} \cdot B\|^2 \end{aligned}$$

$$
\begin{aligned}
&= n\left\{\prod_{i=1}^{n}|\alpha_i|^2\right\}\{(n+p-1)!\|f\|^2\|f^{n-1}\cdot B\|^2\}\\
&= n\left\{\prod_{i=1}^{n}|\alpha_i|^2\right\}\|f\|^2\mathrm{per}(\mathrm{Gr}(f^{\{n-1\}}, f_{n+1}, f_{n+2}, \ldots, f_{n+p}))\\
&= n\|\alpha_1 f\|^2\mathrm{per}(\mathrm{Gr}(\alpha_2 f, \alpha_3 f, \ldots, \alpha_n f, f_{n+1}, f_{n+2}, \ldots, f_{n+p}))\\
&= n\|f_1\|^2\mathrm{per}(\mathrm{Gr}(f_2, f_3, \ldots, f_n, f_{n+1}, f_{n+2}, \ldots, f_{n+p}))\\
&= nm_{11}\mathrm{per}(M(1|1)).
\end{aligned}
\tag{48}
$$

This completes the proof. □

5. Another permanental inequality

Extending our notation somewhat we let $Q_{t,p}^{n}$ denote the set of all strictly increasing sequences from $\{1, 2, \ldots, t\}$ to $\{n+1, n+2, \ldots, n+p\}$. Of course this would mean that $Q_{t,p}^{0}$ is the $Q_{t,p}$ introduced previously. We will translate Corollary 3 into an array of permanental inequalities.

For any finite sequences σ and τ we let $\sigma \cup \tau$ denote the sequence obtained by appending τ to the end of σ; thus, if $\sigma \in Q_{s,n}$ and $\tau \in Q_{t,p}^{n}$, then $\sigma \cup \tau \in Q_{s+t,n+p}$. Recall definition (28) of $\Lambda_{s,t}(A, B)$ where it is assumed that $A \in \mathbf{S}_n(V)$ and $B \in \mathbf{S}_p(V)$. Is is shown in [14] that if $A = f_1 \cdot f_2 \cdots f_n$, and $B = f_{n+1} \cdot f_{n+2} \cdots f_{n+p}$, where each $f_i \in V^*$, and M denotes the Gram matrix, $\mathrm{Gr}(f_1, f_2, \ldots, f_{n+p})$, then

$$
\begin{aligned}
&(n+p)!\Lambda_{s,t}(A, B)\\
&\quad = 1\Big/\left[\binom{n}{s}\right]\binom{p}{t}\Big] \sum_{\alpha,\delta\in Q_{s,n}} \sum_{\beta,\gamma\in Q_{t,p}^{n}} \mathrm{per}(M[\alpha|\delta])\mathrm{per}(M[\beta|\gamma])\mathrm{per}\big(M(\alpha\cup\beta|\delta\cup\gamma)).
\end{aligned}
\tag{49}
$$

Defining $\mathscr{L}_{s,t}^{n,p}(M)$ by

$$
\begin{aligned}
&\mathscr{L}_{s,t}^{n,p}(M)\\
&\quad = \left[1\Big/\binom{n}{s}\binom{p}{t}\right] \sum_{\alpha,\delta\in Q_{s,n}} \sum_{\beta,\gamma\in Q_{t,p}^{n}} \mathrm{per}(M[\alpha|\delta])\mathrm{per}(M[\beta|\gamma])\mathrm{per}(M(\alpha\cup\beta|\delta\cup\gamma)),
\end{aligned}
\tag{50}
$$

for all s and t such that $0 \leqslant s, t \leqslant \kappa$ and $M \in \mathscr{H}_{n+p}$ we obtain the following from Corollary 3.

Corollary 11. *Suppose q is a positive integer and both n and p are positive integers such that $n + p = q$. If $s, u \in \{0, 1, 2, \ldots, n\}$, $t, v \in \{0, 1, 2, \ldots, p\}$, $s \leqslant u$, and $t \leqslant v$, then*

$$\mathscr{L}_{s,t}^{n,p}(M) \geqslant \mathscr{L}_{u,v}^{n,p}(M)$$

for all $q \times q$ positive semi-definite Hermitian matrices M, with equality if M is the $q \times q$ identity matrix.

It is easy to lose the result above in the notation. For example, if $s = 0$ and $t = 0$, then $\mathscr{L}^{n,p}_{s,t}(M) = \mathrm{per}(M)$. Suppose $M = [m_{ij}] \in \mathscr{H}_{n+p}$. One implication of Corollary 11 is that

$$\mathrm{per}(M) \geqslant 1/[np] \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\alpha=n+1}^{n+p} \sum_{\beta=n+1}^{n+p} m_{ij} m_{\alpha\beta} \mathrm{per}(M(\{i, \alpha\}|\{j, \beta\})). \tag{51}$$

To understand this result, imagine that M is partitioned into a 2×2 block matrix $[M_{ij}]$ where M_{11} is $n \times n$ and M_{22} is $p \times p$. The right side of (51) is constructed by choosing an element from M_{11} and an element from M_{22}; we then multiply these two elements by the permanent of the submatrix of M that is obtained by deleting the rows and columns from which our two chosen elements came; finally we sum our products over all possible ways of choosing the initial two elements, and divide by np. There are many other matrix inequalities, some very complex, that also follow from Theorem 3.

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